Phase mixing, induced relaxation, and chaos in one-dimensional dynamical systems

A. Bulatov, B. E. Vugmeister, and H. Rabitz
Department of Chemistry, Princeton University, Princeton, NJ 08544
(Received 12 April 2000; revised manuscript received 29 May 2001; published 24 September 2001)

This paper investigates the mechanism of induced phase mixing, which leads to effective dissipation in classical nonlinear dynamical systems with a fast modulation of the potential. The suggested model can be applied to a classical dynamical description of cold atomic clouds in optical traps. We show that the parametric nonadiabatic modulation of the laser intensity can provide a tool for dynamical control of the effective relaxation in such systems.

DOI: 10.1103/PhysRevE.64.046217 PACS number(s): 05.45.−a

I. INTRODUCTION

This paper explores the effect of induced dissipation [1,2] occurring in nonlinear dynamical systems subjected to a nonadiabatic time-dependent external field. A particular application of this effect is the coherent control of atomic systems in off-resonant dipole optical traps. We show that the phase-space density, the effective relaxation rate, and the energy diffusion coefficients can be controlled by means of the nonstationary external potential.

Our previous paper [3] addressed the effect of increasing the phase-space density (degeneracy degree) for an atomic system in the nonadiabatic regime. In this regime, a specific mechanism of relaxation of the average atomic distribution has to be considered, since the dynamical time scale may not be much longer than the relaxation time. We showed that in the nonadiabatic case the phase-space density can be increased by the nonlinear effects of atomic breathing oscillations due to the instability of such oscillations with respect to the initial conditions. This relaxation mechanism is related to the phenomenon known as ''phase mixing,'' and does not require atomic collisions. This effect opens up the possibility of achieving fast nonadiabatic control of the onset of Bose-Einstein condensation in an atomic cloud loaded into an optical trap.

The experiment of Ref. [4] reported the optical trapping of a Bose-Einstein condensate. An important experimental observation in Ref. [4] was that condensation occurred in the optical trap even when it was loaded with noncondensed magnetically trapped atoms. The authors suggested that this phenomenon is related to the effect of increasing the phase-space density of the atomic system with the adiabatically changing shape of the confining potential [5]. We showed that a similar effect of phase compression should also take place for a parametric modulation of the anharmonic optical potential [3]. One needs to take into account some relaxation mechanism in order to circumvent the conservation of phase volume imposed by the Liouville theorem. In Ref. [5], it was assumed that this relaxation mechanism was provided by atomic collisions.

In this paper, we show that an induced dissipation mechanism should exist in nonlinear dynamical systems subject to a suitable time-dependent external field. This effect is related to the recently experimentally observed [6] and theoretically predicted [7] echo effect of squeezing oscillations in optical traps, occurring due to the partial reverse of dephasing in anharmonic potentials. The echo effect was achieved by a parametric modulation of the potential by sufficiently short pulses. As we show below, a complex time structure of the modulation of the potential may cause effective dissipation in an atomic system depending on the parameters of the modulation.

We show that the relaxation rates and effective energy diffusion coefficients associated with this mechanism can be controlled by adjusting the time profile of the laser intensity. One should note that the discussed effect should generally occur in any dynamical system with a sufficiently low initial dissipation.

II. MODEL

Following the experimental situation described earlier [4], we assume that after an initial cooling to temperature T, the atoms are transferred into an optical trap with a large detuning, and that they are located near the minimum of the optical potential. The temperature T is assumed to be higher than the Bose-Einstein condensation transition temperature T_c. Since the detuning is large, the dissipation effects are small and will be disregarded. We consider the case of temperatures T much larger than the energy $\hbar\omega_0$ of the atomic oscillations at the bottom of the potential. Consequently, no quantum dynamical effects are taken into account. On the other hand, the temperature should be low enough to insure that the atoms do not escape from the potential well. The effective potential is approximated as a one-dimensional oscillator with weak anharmonicity and time-dependent frequency given by

$$U(x,t) \approx \frac{m\omega(t)^2 x^2}{2} + \frac{m\eta(t)x^4}{4}, \tag{1}$$

with harmonic frequency $\omega(t)$, anharmonicity parameter $\eta(t)$, and atomic mass m. Note that nonlinear dynamical systems with dimensionality $D>1$ can exhibit a chaotic behavior even in time-independent fields. In the case $D=1$ and with static external fields, the chaotic behavior is absent due to the complete integrability of one-dimensional (1D) dynamics systems [8]. However, chaotic behavior can occur in 1D systems with time-dependent external potentials, also referred to as 3/2D dynamical systems [1]. The 3/2D situation is a limiting case when the chaotic regime may exist for

certain parameters of the dynamical system and a certain time modulation of the potential. However, this case is analytically more tractable.

The time evolution of the distribution function is described by the Liouville equation

$$\frac{\partial \rho}{\partial t}+\frac{p}{m}\frac{\partial \rho}{\partial x}-(m\omega^2 x+m\eta x^3)\frac{\partial \rho}{\partial p}=0. \tag{2}$$

We introduce the phase space polar coordinates I and Θ by

$$\begin{aligned}\frac{p^2}{2m}&=I\omega\sin^2\Theta,\\ \frac{m\omega^2x^2}{2}+\frac{m\eta x^4}{4}&=I\omega\cos^2\Theta.\end{aligned} \tag{3}$$

In terms of the coordinates given by Eq. (3), the equations of motion are

$$\begin{aligned}\frac{d\Theta}{dt}&=\Omega(I)-\frac{1}{2\omega}\frac{d}{dt}[\omega(t)]\sin(2\Theta),\\ \frac{dI}{dt}&=\frac{1}{\omega}\frac{d}{dt}[\omega(t)]I\cos(2\Theta),\end{aligned} \tag{4}$$

In the limit of small anharmonicity the approximation

$$\Omega(I)\approx\omega\left(1+\frac{3}{4}\frac{\eta}{m\omega^2}I\right)$$

is valid.

In terms of the action-angle variables, the Liouville equation takes the form [2,1]

$$i\frac{\partial \rho}{\partial t}=(\hat{L}_0+\hat{L}_1)\rho, \tag{5}$$

where the Hermitian operators $\hat{L}_0$ and $\hat{L}_1$ are given by

$$\begin{aligned}\hat{L}_0&=i\Omega(I,t)\frac{\partial}{\partial\Theta},\\ \hat{L}_1&=(-i)\left(\frac{\partial V}{\partial\Theta}\frac{\partial}{\partial I}-\frac{\partial V}{\partial I}\frac{\partial}{\partial\Theta}\right),\\ V(I,\Theta)&=\frac{1}{2\omega(t)}\frac{d}{dt}(\omega(t))I\sin(2\Theta).\end{aligned} \tag{6}$$

Following Ref. [2], we expand the distribution function in a Fourier series in the angle Θ. Due to the periodicity of the distribution function with respect to Θ, the expansion is given by

$$\rho(I,\Theta,t)=\sum_{n=-\infty}^{+\infty}\rho_n(I,t)\exp(in\Theta), \tag{7}$$

with the Fourier components ρ_n obeying equations of motion in a form familiar from time-dependent perturbation theory in quantum mechanics,

$$i\frac{\partial \rho_n}{\partial t}=\sum_k\langle k|\hat{L}_0+\hat{L}_1|n\rangle\rho_k, \tag{8}$$

where the matrix elements of the Liouville operator are given by $\langle k|\hat{L}|n\rangle=(1/2\pi)\int_0^{2\pi}d\Theta\exp(-ik\Theta)\hat{L}\exp(in\Theta)$. Expanding the non-adiabatic contribution as

$$V(I,\Theta,t)=\sum_{n=-\infty}^{+\infty}V_n(I,t)\exp(in\Theta), \tag{9}$$

we obtain the matrix elements of the Liouville operator in the form

$$\begin{aligned}\langle n|\hat{L}_0|k\rangle&=\Omega n\,\delta_{kn},\\ \langle n|\hat{L}_1|k\rangle&=(n-k)V_{n-k}(I)\frac{\partial}{\partial I}-k\frac{\partial V_{n-k}(I)}{\partial I}.\end{aligned} \tag{10}$$

Note that the matrix elements of the Liouville operator with respect to the angle variables are still operators with respect to the action variables [1]. Introducing the new variables ψ_n as

$$\begin{aligned}\rho_n(I,t)&=\psi_n(I,t)\exp(in\Phi),\\ \Phi(t)&=\int_0^t dt'\Omega(t'),\end{aligned} \tag{11}$$

the equations of motion given by Eq. (8) reduce to

$$i\frac{\partial \psi_n}{\partial t}=\sum_{k\neq n}\langle k|\hat{L}_1|n\rangle\psi_k\exp[-i(n-k)\Phi]. \tag{12}$$

Substituting the expression for the nonadiabatic perturbation operator V from Eq. (6) into Eq. (10), we obtain

$$\begin{aligned}\langle n|\hat{L}_1|n+2\rangle&=2V_2(I)\frac{\partial}{\partial I}-(n+2)\frac{\partial V_2(I)}{\partial I},\\ \langle n|\hat{L}_1|n-2\rangle&=-2V_2^*(I)\frac{\partial}{\partial I}-(n-2)\frac{\partial V_2^*(I)}{\partial I},\end{aligned} \tag{13}$$

with $V_2=V_{-2}^*=-i[1/4\omega(t)](d/dt)\omega(t)$. Making use of Eqs. (12) and (13), we obtain a set of dynamical equations for the distribution function with appropriate initial conditions.

III. INDUCED RELAXATION IN 3/2D DYNAMICAL SYSTEMS

We analyze the nonequilibrium energy distribution in the case when fast squeezing oscillations are present. As shown in Refs. [3,5], the average phase-space volume is not conserved for adiabatic modulation of the optical potential. This effect is due to relaxation which does not have to be speci-

fied in this regime. In the nonadiabatic regime, a similar effect is expected, but in this case a specific relaxation mechanism must be present. We consider a collisionless gas of atoms in an anharmonic optical trap with a fast parametric modulation of the optical potential, and show that there exists a relaxation mechanism arising from instabilities of the atomic trajectories with respect to the initial conditions. Such a mechanism is related to phase mixing [1].

As shown above, in a potential with quartic anharmonicity the Liouville equation for the nonequilibrium distribution function can be presented in action-angle variables in the form of coupled dynamical equations for the different order Fourier components of the distribution function or coherences. In order to solve the equations of motion, the infinite chain given by Eq. (12) has to be broken at some point. This is analogous to the procedure of breaking the high-order correlations which is necessary to obtain closed-form kinetic equations [2].

We are interested in the time evolution of the coherences averaged over a sufficiently long time interval. Due to the nonadiabatic terms, the action acquires small but rapidly oscillating contributions as $I=\langle I\rangle+\delta I$. Correspondingly, the phase also acquires such contributions as $\Phi=\langle\Phi\rangle+\delta\Phi$. This leads to a dephasing and decay of the high-order correlations averaged over the ensemble. The dephasing originates from the rapidly oscillating phases on the right-hand side of Eq. (11), which lead to a decay after averaging as

$$\exp[in(\langle\Phi\rangle+\delta\Phi)]\rangle\approx\left\langle \exp\left(in\int_0^t dt'\,\omega[1+\varepsilon(\langle I\rangle + \delta I(t'))]\right)\right\rangle, \tag{14}$$

where $\varepsilon=3/4(\eta/m\omega^2)\equiv$const, and the averaging is performed over fast fluctuations of the action. It is known [1] that if the distribution in phase space is sufficiently smooth, the correlations $\langle\delta I(t)\,\delta I(t+\tau)\rangle$ decay rapidly in τ due to the dephasing. This condition is naturally satisfied in our quasiequilibrium system [3,7]. In the limit $\tau\gg\tau_c$, where τ_c is the typical correlation time, the correlators can be approximated as $\langle\delta I(t)\,\delta I(t')\rangle\approx\langle\delta I^2\rangle\tau_c\delta(t-t')$. Note that the dephasing can be partially reversed by means of a short-pulse modulation of the potential, leading to an echo effect [7]. As we show below, a more complex time dependence of the modulation leads to an effective dissipation. After averaging over a time interval greater than the correlation time, we obtain

$$\left\langle \exp\left[in\epsilon\int_0^t dt'\,\delta I(t')\right]\right\rangle \approx\exp\left[-\frac{1}{2}n^2\varepsilon^2\int_0^t dt'\int_0^{t'} dt''\langle\delta I(t')\,\delta I(t'')\rangle\right], \tag{15}$$

which leads to

$$\langle\exp[in\Phi]\rangle\approx\exp[in\langle\Phi\rangle]\exp\left(-\frac{t}{\tau_r}\right), \tag{16}$$

with $\tau_r^{-1}=1/2\tau_c n^2\epsilon^2\ \langle\delta I^2\rangle$. To estimate a lower bound of the relaxation time, we assume that $\tau_r=\tau_c$, and obtain

$$\tau_r^{-1}=\frac{1}{\sqrt{2}}n\varepsilon\sqrt{\langle\delta I^2\rangle}. \tag{17}$$

The estimate $\langle\delta I^2\rangle\approx(1/\omega^2)\langle\delta E^2\rangle=T^2/\omega^2$ yields

$$\rho_r\sim\frac{1}{\omega}\frac{1}{nK}, \tag{18}$$

where $K=\eta T/m\omega^4$, according to our previous results [3].

In order to simplify the calculations, we will break this chain at the fourth order Fourier component or coherency in Eq. (12). Then we obtain

$$\begin{aligned} i\frac{\partial\psi_0}{\partial t}&=-\frac{\partial}{\partial I}(2U_2^*\psi_2-2U_2\psi_2^*),\\ i\frac{\partial\psi_2}{\partial t}&=2U_2\frac{\partial}{\partial I}\psi_0+2U_2^*\frac{\partial}{\partial I}\psi_4-4\frac{\partial}{\partial I}(U_2^*\psi_4),\\ i\frac{\partial\psi_4}{\partial t}&\approx 4U_2\frac{\partial}{\partial I}\psi_2-2\frac{\partial}{\partial I}(U_2\psi_2), \end{aligned} \tag{19}$$

where $U_{n-k}=V_{nk}\exp[-i(n-k)\Phi]$. In the last expression of Eqs. (19), we have neglected the coupling of the fourth coherency to higher-order coherences, and thus obtained a closed set of equations for the first four coherences. Note that the truncation procedure is standard [2], and is necessary in order to obtain a closed set of equations for the lower correlators. Essentially, it is justified by the assumption that higher-order correlations decay faster than the lower-order ones. Expressing ψ_4 in terms of ψ_2 and substituting back into Eq. (19), we obtain a closed set of equations for ψ_0 and ψ_2 in the forms

$$\begin{aligned} \frac{\partial\psi_0}{\partial t}&=\frac{\partial}{\partial I}\left(D_0(I)\frac{\partial}{\partial I}\psi_0\right),\\ \frac{\partial\psi_2}{\partial t}&=-2iU_2\frac{\partial}{\partial I}\psi_0+\frac{\partial}{\partial I}\left(D_2(I)\frac{\partial}{\partial I}\psi_2\right)-\Gamma_2\psi_2, \end{aligned} \tag{20}$$

where

$$\begin{aligned} D_0&=8\tau_r|U_2|^2,\\ D_2&=4\tau_r|U_2|^2,\\ \Gamma_2&=8\tau_r\left|\frac{\partial U_2}{\partial I}\right|^2. \end{aligned} \tag{21}$$

Note that the relaxation time τ_r enters Eq. (21) due to the exponential decay of the phase given by Eq. (20). From Eq. (20), it follows that the second-order coherency $\psi_2(t)$ decays with a rate Γ_2 given by Eq. (21). The higher-order coher-

ences decay even faster, since they depend on higher powers of the relative phase, as shown above. Due to this behavior, the angular dependence of the distribution function decays, and the average distribution function becomes only a function of energy. This is analogous to what happens as a result of virtually any relaxation mechanism in the adiabatic regime (but for a nonaveraged distribution). Therefore, the average phase-space distribution experiences relaxation due to the non-adiabatic phase-mixing effects considered above.

The relaxation rate and the effective diffusion coefficient for zero-order coherency are determined by the time-dependent external potential acting on the dynamical system. Therefore, the relaxation process depends on the time profile of the external potential acting on the dynamical system, and can be effectively manipulated by changing this time profile.

IV. CONCLUSION

This paper investigated the effect of induced dissipation in nonlinear dynamical systems subjected to nonadiabatic time-dependent external fields. Applications of interest might be to coherent control of atomic systems in off-resonant dipole optical traps. This work showed that the phase-space density, the effective relaxation rate, and the energy diffusion coefficients can be controlled [9] by means of a nonstationary external potential.

ACKNOWLEDGMENT

The authors acknowledge support from the Department of Defense and the National Science Foundation.

[1] G. M. Zaslavsky, *Chaos in Dynamic Systems* (Harwood, New York, 1985).

[2] I. Prigogine, *Non-equilibrium Statistical Mechanics* (Wiley, New York, 1962).

[3] A. Bulatov, B. E. Vugmeister, and H. Rabitz, Phys. Rev. A **60**, 4875 (1999).

[4] D. M. Stamper-Kurn, M. R. Andrews, A. P. Chikkatur, S. Inouye, H.-J. Miesner, J. Stenger, and W. Ketterle, Phys. Rev. Lett. **80**, 2027 (1998).

[5] P. W. H. Pinske, A. Mosk, M. Weidemuller, M. W. Reynolds, T. W. Hijmans, and J. T. M. Walraven, Phys. Rev. Lett. **78**, 990 (1997).

[6] F. B. J. Buchkremer, R. Dumke, H. Levsen, G. Birkl, and W. Ertmer, Phys. Rev. Lett. **85**, 3121 (2000).

[7] A. Bulatov, A. Kuklov, B. E. Vugmeister, and H. Rabitz, Phys. Rev. A **57**, 1 (1998).

[8] N. N. Bogoliubov and Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Gordon and Breach, New York, 1961).

[9] S. Shi and H. Rabitz, J. Chem. Phys. **92**, 364 (1990).